

ÉTALE HOMOTOPY TYPES OF MODULI STACKS OF POLARISED ABELIAN SCHEMES

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ABSTRACT. We determine the Artin-Mazur étale homotopy types of moduli stacks of polarised abelian schemes using transcendental methods and derive some arithmetic properties of the étale fundamental groups of these moduli stacks. Finally we analyse the Torelli morphism between the moduli stacks of algebraic curves and principally polarised abelian schemes from an étale homotopy point of view.

INTRODUCTION

The use of the Artin-Mazur machinery of étale homotopy theory [AM] for Deligne-Mumford stacks was pioneered by Oda [O] following ideas of Grothendieck [G] in order to study arithmetic homotopy types of moduli stacks of algebraic curves and to relate them to geometric representations of the absolute Galois group $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$. Let $\mathcal{M}_{g,n}$ be the moduli stack of smooth algebraic curves of genus g with n distinct ordered points and $2g + n > 2$. Oda determined the Artin-Mazur étale homotopy type of the moduli stack $\mathcal{M}_{g,n} \otimes \bar{\mathbb{Q}}$ over the big étale site of schemes over \mathbb{Q} . Using transcendental methods by comparing it with the complex analytic situation Oda showed that the étale homotopy type $\mathcal{M}_{g,n} \otimes \bar{\mathbb{Q}}$ is given as the profinite Artin-Mazur completion of the Eilenberg-MacLane space $K(\mathrm{Map}_{g,n}, 1)$, where $\mathrm{Map}_{g,n}$ is the Teichmüller modular or mapping class group of compact Riemann surfaces of genus g with n punctures. Oda's results allow to analyse geometric Galois actions as they give rise to a short exact sequence relating the absolute Galois group $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$, the étale fundamental group $\pi_1^{et}(\mathcal{M}_{g,n} \otimes \bar{\mathbb{Q}}, x)$ of the moduli stack of algebraic curves over the rationals \mathbb{Q} and the profinite completion $\mathrm{Map}_{g,n}^\wedge$ of the mapping class groups $\mathrm{Map}_{g,n}$ (cf. [Ma], [SL], [Mk]).

In [FN] the authors followed up on the theme of Oda and determined the étale homotopy types of related moduli stacks of algebraic curves

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with prescribed symmetries including moduli stacks of hyperelliptic curves. In a recent article Ebert and Giansiracusa [EG] determined also the étale homotopy types of the Deligne-Mumford compactifications $\tilde{\mathcal{M}}_{g,n}$ which classify stable algebraic curves of genus g with n distinct ordered points and are compactifications of the moduli stacks $\mathcal{M}_{g,n}$.

In this article we will now analyse étale homotopy types of moduli stacks \mathcal{A}_D of abelian schemes with polarisations of a general given type D and related moduli stacks $\mathcal{A}_{D,[N]}$ of polarised abelian schemes with level structures. Special cases include the moduli stack \mathcal{A}_g of principally polarised abelian varieties and the moduli stack \mathcal{M}_{ell} of elliptic curves. Moduli stacks of abelian schemes and their compactifications play a fundamental role in algebraic geometry and number theory.

It turns out that a similar theorem like that of Oda for the moduli stacks of algebraic curves holds with some modifications also for the moduli stacks \mathcal{A}_D and $\mathcal{A}_{D,[N]}$ of polarised abelian schemes with and without level structures.

More precisely, let $g \geq 1$ be a positive integer and $D = (d_1, d_2, \dots, d_g)$ be a tuple of integers such that $d_1 | d_2 | \dots | d_g$. The moduli stack \mathcal{A}_D of abelian schemes with polarisations of type D is known to be an algebraic Deligne-Mumford stack. It has also a complex analytic uniformisation \mathcal{A}_D^{an} given as a complex analytic Deligne-Mumford quotient stack or complex analytic orbifold of the form $\mathcal{A}_D^{an} = [\mathfrak{H}_g / \mathrm{Sp}_D(\mathbb{Z})]$. Here \mathfrak{H}_g is the Siegel upper half space of genus g , which carries a natural action of the discrete group $\mathrm{Sp}_D(\mathbb{Z}) = \{\gamma \in \mathrm{Mat}(2g, \mathbb{Z}) \mid {}^t \gamma J_D \gamma = J_D\}$, where the matrix J_D is given as

$$J_D = \begin{pmatrix} 0 & D \\ -D & 0 \end{pmatrix},$$

and D is the diagonal matrix with entries (d_1, \dots, d_g)

In order to determine the étale homotopy type $\{\mathcal{A}_D \otimes \bar{\mathbb{Q}}\}_{et}$ we compare the algebraic with the associated complex analytic situation. As the Siegel upper half space \mathfrak{H}_g is a contractible topological space this allows to determine the classical homotopy type of the stack \mathcal{A}_D^{an} and using a general comparison theorem comparing étale and classical homotopy types essentially due to Artin-Mazur [AM] we can derive our main result

Theorem. Let $g \geq 1$ be a positive integer and $D = (d_1, d_2, \dots, d_g)$ be a tuple of integers with $d_1 | d_2 | \dots | d_g$. There is a weak homotopy equivalence of pro-simplicial sets

$$\{\mathcal{A}_D \otimes \bar{\mathbb{Q}}\}_{et}^\wedge \simeq K(\mathrm{Sp}_D(\mathbb{Z}), 1)^\wedge.$$

where $^{\wedge}$ denotes Artin-Mazur profinite completion. In particular for principal polarisations we have

$$\{\mathcal{A}_g \otimes \bar{\mathbb{Q}}\}_{et}^{\wedge} \simeq K(\mathrm{Sp}(2g, \mathbb{Z}), 1)^{\wedge}.$$

An important special case of \mathcal{A}_g is given by the moduli stack \mathcal{M}_{ell} of elliptic curves. Here $g = 1$ and we have $\{\mathcal{M}_{ell} \otimes \bar{\mathbb{Q}}\}_{et}^{\wedge} \simeq K(\mathrm{SL}(2, \mathbb{Z}), 1)^{\wedge}$. The arithmetic properties of the Deligne-Mumford stack \mathcal{M}_{ell} were first studied by Deligne and Rapoport [DR] and it plays a fundamental role also in the systematic construction of elliptic cohomology theories [Lu].

We also derive similar results for the moduli stacks $\mathcal{A}_{D,[N]}$ which are modifications of our main result using appropriate congruence subgroups $\Gamma_D(N)$ of $\mathrm{Sp}_D(\mathbb{Z})$ which reflect the particular level N structures.

From these general results we also obtain short exact sequences involving the étale fundamental groups of the moduli stacks \mathcal{A}_D and $\mathcal{A}_{D,[N]}$ relating them to the profinite completions of the discrete groups $\mathrm{Sp}_D(\mathbb{Z})$ and $\Gamma_D(N)$ and the absolute Galois group $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$. In this way we extend the results of Oda to other interesting moduli stacks which allow for further interpretations of geometric Galois actions. As a bonus we can directly relate the étale homotopy types of the moduli stacks \mathcal{M}_g of algebraic curves with the étale homotopy types of the moduli stacks \mathcal{A}_g of principally polarised abelian schemes via the stacky Torelli morphism $j : \mathcal{M}_g \rightarrow \mathcal{A}_g$ between these moduli stacks, which is induced by taking Jacobians of algebraic curves. We also make similar observations again for the moduli stacks $\mathcal{M}_{g,[N]}$ and $\mathcal{A}_{g,[N]}$ which take into account the level structures.

The paper is organized as follows: In the first section we recall aspects of the general theory of moduli of abelian schemes including polarisations and level structures and introduce the respective moduli stacks. In the next section we define Artin-Mazur homotopy types of Deligne-Mumford stacks and compare the étale and classical homotopy types in the algebraic and complex analytic context. In the third section we determine the étale homotopy types of the moduli stacks of abelian schemes over the algebraic closure of the rational numbers. And in the final section we relate the étale homotopy types of the moduli stacks of algebraic curves and abelian schemes with principal polarisations by analysing the stacky Torelli morphism.

1. MODULI STACKS OF ABELIAN SCHEMES WITH POLARISATIONS

We will recollect in this section the main aspects of the theory of abelian schemes with polarisations and their moduli following Mumford [M2], Chai [C], Faltings-Chai [FC] and Mumford-Fogarty-Kirwan

[MFK]. For the general theory of algebraic stacks we refer to the book of Laumon and Moret-Bailly [LMB] and the online Stack Project [Stacks]. Other good sources are [DM], [V], [Go] and [Fa].

Definition 1.1. Let S be a base scheme. An *abelian scheme* A/S is a proper smooth group scheme $\pi : A \rightarrow S$ over S with connected geometric fibers.

Being a group scheme, an abelian scheme A/S is equipped with the following structure morphisms:

- (1) a group law or multiplication morphism $\mu : A \times_S A \rightarrow A$,
- (2) a unit section morphism $e : S \rightarrow A$,
- (3) an inverse morphism $i : A \rightarrow A$,

subject to the usual axioms for being an abstract group.

It turns out, that in fact any abelian scheme A/S is a commutative group scheme of finite presentation (cf. [FC], [MFK]).

Let A/S be an abelian scheme, \mathcal{L} be an invertible sheaf on A and

$$p_1, p_2 : A \times_S A \rightarrow A$$

be the projections to the first and second factor respectively. Then it follows that the induced sheaf

$$\mu^* \mathcal{L} \otimes p_1^* \mathcal{L}^{-1} \otimes p_2^* \mathcal{L}^{-1}$$

is an invertible sheaf on the fiber product $A \times_S A$. Regarding $A \times_S A$ as a scheme over A via p_2 we get therefore an S -morphism

$$\lambda(\mathcal{L}) : A \rightarrow \mathrm{Pic}^0(A/S).$$

Here $\mathrm{Pic}^0(A/S)$ is an open subspace and the neutral component of the abelian algebraic space $\mathrm{Pic}(A/S)$ representing the relative Picard functor $\mathfrak{Pic}_{A/S}$, which corresponds to the subfunctor $\mathfrak{Pic}_{A/S}^0$. $\mathrm{Pic}^0(A/S)$ is in fact an abelian scheme by a theorem of Raynaud [FC, Theorem 1.9] called the *dual abelian scheme* of A/S . By the theorem of the cube, the S -morphism $\lambda(\mathcal{L})$ is actually a group homomorphism respecting the group structures of A/S and $\mathrm{Pic}^0(A/S)$ (cf. [MFK], [FC], [GeN]). We now define:

Definition 1.2. Let S be a base scheme. A *polarisation* of an abelian scheme A/S is an S -homomorphism of group schemes

$$\lambda : A/S \rightarrow \mathrm{Pic}^0(A/S)$$

such that for each geometric point s of S the induced homomorphism

$$\lambda_s : A_s \rightarrow \mathrm{Pic}^0(A_s)$$

is of the form $\lambda_s = \lambda(\mathcal{L}_s)$ for some ample invertible sheaf \mathcal{L}_s on A_s . The kernel of λ has rank r^2 for some locally constant positive integer valued function r on S and r is called the degree of λ . λ is called a *principal polarisation* if it is an isomorphism.

If $\lambda : A/S \rightarrow \text{Pic}^0(A/S)$ is a polarisation of an abelian scheme A/S , then its degree r is constant on each connected component and λ is a finite and faithfully flat morphism, i.e an isogeny. In fact, any polarisation of an abelian scheme A/S is a symmetric isogeny $\lambda : A/S \rightarrow \text{Pic}^0(A/S)$, which locally for the étale topology of S is of the form $\lambda(\mathcal{L})$ for some ample line bundle \mathcal{L} on A/S .

If the base scheme S is connected with residue characteristic prime to the degree of λ , then there exists integers $d_1|d_2|\dots|d_g$, where g is the relative dimension of A/S , such that for any geometric point s of S we have an isomorphism of abelian groups (cf. [GeN])

$$\ker \lambda_s \cong (\mathbb{Z}/d_1\mathbb{Z} \times \mathbb{Z}/d_2\mathbb{Z} \times \dots \times \mathbb{Z}/d_g\mathbb{Z})^2.$$

The tuple $D = (d_1, d_2, \dots, d_g)$ is called the *type of the polarisation* λ of the abelian scheme A/S . If λ is a principal polarisation, its type is just $D = (1, 1, \dots, 1)$ and especially $d_g = 1$.

Let $V = \mathbb{Z}^{2g}$ and let $\phi_D : V \times V \rightarrow \mathbb{Z}$ be a symplectic pairing, such that on the standard ordered basis $\{e_i\}_{i=1,\dots,2g}$ we have $\phi_D(e_i, e_j) = 0$ if $i, j \leq g$ or $i, j \geq g$ and $\phi_D(e_i, e_{j+g}) = \delta_{i,j} \cdot d_i$ if $i, j \leq g$. The group $\text{CSp}_D = \text{CSp}(V, \phi_D)$ is the reductive algebraic group over $\text{Spec}(\mathbb{Z})$ of symplectic similitudes of V with respect to the form ϕ_D and $\nu : \text{CSp}_D \rightarrow \mathbb{G}_m$ is the associated multiplier character. The kernel $\text{Sp}_D = \ker \nu$ is the symplectic group Sp_D of all transformations of V preserving the form ϕ_D . In the special case that $d_g = 1$ we get the familiar groups CSp_g and Sp_{2g} (cf. [MO]).

We will now define the moduli stack of polarised abelian schemes, which will be the main object of our studies.

Definition 1.3. Let $g \geq 1$ be a positive integer and $D = (d_1, d_2, \dots, d_g)$ be a tuple of integers with $d_1|d_2|\dots|d_g$. Let $(\text{Sch}/\mathbb{Z}[(d_g)^{-1}])$ be the category of schemes over $\text{Spec}(\mathbb{Z}[(d_g)^{-1}])$ and \mathcal{A}_D be the category fibred in groupoids over $(\text{Sch}/\mathbb{Z}[(d_g)^{-1}])$ defined by its groupoid of sections $\mathcal{A}_D(S)$ as follows:

For a scheme S over $\text{Spec}(\mathbb{Z}[(d_g)^{-1}])$, let $\mathcal{A}_D(S)$ be the groupoid whose objects are the pairs $(A/S, \lambda)$, where A/S is an abelian scheme over S and

$$\lambda : A/S \rightarrow \text{Pic}^0(A/S)$$

a polarisation of type D . The morphisms of $\mathcal{A}_D(S)$ are S -isomorphisms of abelian schemes compatible with polarisations of type D , i.e. a morphism between $(A/S, \lambda)$ and $(A'/S, \lambda')$ is a homomorphism of abelian schemes $\varphi : A/S \rightarrow A'/S$ such that $\varphi^*(\lambda') = \lambda$.

If we give the category $(Sch/\mathbb{Z}[(d_g)^{-1}])$ the étale topology, then \mathcal{A}_D is a stack over the big étale site $(Sch/\mathbb{Z}[(d_g)^{-1}])_{et}$, called the *moduli stack of abelian schemes with polarisations of type D* .

If $d_g = 1$, then \mathcal{A}_D is the *moduli stack \mathcal{A}_g of principally polarised abelian schemes of relative dimension g* over $(Sch/\mathbb{Z})_{et}$.

It turns out that the moduli stacks \mathcal{A}_D behave very nicely. In fact, they are algebraic stacks in the sense of Deligne-Mumford [DM] on the big étale site $(Sch/\mathbb{Z}[(d_g)^{-1}])_{et}$.

Theorem 1.4. Let $g \geq 1$ be a positive integer and $D = (d_1, d_2, \dots, d_g)$ be a tuple of positive integers with $d_1 | d_2 | \dots | d_g$. The moduli stack \mathcal{A}_D is a smooth quasi-projective Deligne-Mumford stack of finite type over $\text{Spec}(\mathbb{Z}[(d_g)^{-1}])$.

In particular for $d_g = 1$, the moduli stack \mathcal{A}_g is a quasi-projective smooth Deligne-Mumford stack of finite type over $\text{Spec}(\mathbb{Z})$ of relative dimension $g(g+1)/2$.

Proof. That \mathcal{A}_D is a Deligne-Mumford stack is a variation of the proof of the same property for the stack \mathcal{A}_g in [FC, 4.11], [LMB, (4.6.3)] for moduli of principally polarised abelian schemes using polarisations of general type D and the other assertions follow by general GIT arguments (cf. [GeN, Section 2.3]). \square

We will be mainly interested in the restriction of these moduli stacks \mathcal{A}_D and \mathcal{A}_g to the category of schemes over the field of rational numbers \mathbb{Q} and over its algebraic closure $\bar{\mathbb{Q}}$.

Let us denote by \mathcal{A}_D^{an} the complex analytification or the uniformisation of the moduli stack \mathcal{A}_D . It is a complex analytic Deligne-Mumford stack, i.e. a complex analytic orbifold over the big site $(AnSp)_{cl}$ of complex analytic spaces with the classical topology of local isomorphisms (cf. [T], [FN, Section 2.3] for a general discussion about the complex analytification of Deligne-Mumford stacks).

In fact, the moduli stack \mathcal{A}_D^{an} is a complex analytic quotient stack of the form

$$\mathcal{A}_D^{an} = [\mathfrak{H}_g / \text{Sp}_D(\mathbb{Z})]$$

where \mathfrak{H}_g is the *Siegel upper half space of genus g* given as

$$\mathfrak{H}_g = \{\Omega \in \text{Mat}(g, \mathbb{C}) \mid \Omega = {}^t\bar{\Omega}, \text{Im}(\Omega) > 0\}$$

and where $\mathrm{Sp}_D(\mathbb{Z}) = \{\gamma \in \mathrm{Mat}(2g, \mathbb{Z}) \mid {}^t\gamma J_D \gamma = J_D\}$ with

$$J_D = \begin{pmatrix} 0 & D \\ -D & 0 \end{pmatrix},$$

D is the diagonal matrix with entries (d_1, \dots, d_g) and the action of $\mathrm{Sp}_D(\mathbb{Z})$ on \mathfrak{H}_g is given as follows:

$$(1.1) \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \Omega = (a\Omega + bD)(D^{-1}c\Omega + D^{-1}dD)^{-1}.$$

In fact, using Riemann's bilinear relations (see e.g. [BL]) one can prove that \mathfrak{H}_g parametrises isomorphism classes of abelian varieties with a polarisation of type D and with a choice of a symplectic basis.

More precisely, once we fix D , to a point $\Omega \in \mathfrak{H}_g$ one associates the triple $(A_\Omega = \mathbb{C}^g/\Lambda_\Omega, H_\Omega, \{\lambda_1, \dots, \lambda_{2g}\})$, where $\Lambda_\Omega = \Omega \cdot \mathbb{Z}^g + D \cdot \mathbb{Z}^g$, $H_\Omega = (\mathrm{Im}(\Omega))^{-1}$ is the Hermitian form on \mathbb{C}^g which gives the polarisation and $\{\lambda_1, \dots, \lambda_{2g}\}$ are the columns of the $(g \times 2g)$ -period matrix $(\Omega \ D)$. One immediately checks that with respect to this basis $\mathrm{Im}(H_\Omega)|_{\Lambda_\Omega \times \Lambda_\Omega}$ is given by the matrix $\mathrm{Im}({}^t(\Omega \ D)(\mathrm{Im}(\Omega))^{-1}(\overline{\Omega \ D})) = J_D$, so the basis $\{\lambda_1, \dots, \lambda_{2g}\}$ is symplectic. Now, if one wants to parametrise isomorphism classes of abelian varieties with a polarisation of type D , one has to take the quotient of \mathfrak{H}_g by the action of the group $\mathrm{Sp}_D(\mathbb{Z})$. This action is properly discontinuous (cf. [BL]), hence the quotient $\mathfrak{H}_g/\mathrm{Sp}_D(\mathbb{Z})$ has the structure of a complex analytic orbifold.

In the case $d_g = 1$ we get the complex analytic Deligne-Mumford stack \mathcal{A}_g^{an} , which is given as the complex analytic quotient stack or orbifold

$$\mathcal{A}_g^{an} = [\mathfrak{H}_g/\mathrm{Sp}(2g, \mathbb{Z})]$$

where $\mathrm{Sp}(2g, \mathbb{Z})$ is the symplectic group

$$\mathrm{Sp}(2g, \mathbb{Z}) = \{\gamma \in \mathrm{Mat}(2g, \mathbb{Z}) \mid {}^t\gamma J \gamma = J\}$$

with

$$J = \begin{pmatrix} 0 & I_g \\ -I_g & 0 \end{pmatrix},$$

The action (1.1) becomes:

$$\begin{aligned} \mathrm{Sp}(2g, \mathbb{Z}) \times \mathfrak{H}_g &\rightarrow \mathfrak{H}_g \\ \left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}, \Omega \right) &\mapsto (a\Omega + b) \cdot (c\Omega + d)^{-1}. \end{aligned}$$

and the universal family of principally polarised abelian varieties over the Siegel upper half space \mathfrak{H}_g has as fiber over a point $\Omega \in \mathfrak{H}_g$ $A_\Omega = \mathbb{C}^g/(\Omega\mathbb{Z}^g + \mathbb{Z}^g)$, with the polarisation given by $(\mathrm{Im}(\Omega))^{-1}$.

We also like to look at rigidified moduli spaces of abelian schemes using level structures. We recollect here the basic definitions and properties (cf. [FC], [GeN], [BL], [C] and [MO]).

As before, let $g, N \geq 1$ be integers and $D = (d_1, d_2, \dots, d_g)$ be a tuple of positive integers with $d_1 | d_2 | \dots | d_g$ and $\gcd(d_g, N) = 1$. Let A/S be an abelian scheme of relative dimension g over a base scheme S and

$$\lambda : A/S \rightarrow \text{Pic}^0(A/S)$$

a polarisation of type $D = (d_1, d_2, \dots, d_g)$. We have the endomorphism of schemes $[N] : A \rightarrow A$ given by multiplication with N and we will denote its kernel by $A[N] = \ker[N]$. $A[N]$ is a locally free finite group scheme of rank N^{2g} . We also have the Weil pairing

$$e_N^D : A[N] \times A[N] \rightarrow \mu_N,$$

where μ_N is the group of N -th roots of unity. The morphism e_N^D is a non-degenerate bilinear form, which is also symplectic meaning that $e_N^D(x, x) = 1$ for all schemes U over the base S and all U -valued points $x \in A[N](U)$. If in addition N is invertible in $\Gamma(S, \mathcal{O}_S)$, the group scheme $A[N]$ is also étale over S . Now let $\mathbb{Z}[\zeta_N] = \mathbb{Z}[t]/(\Phi_N)$, where $\Phi_N \in \mathbb{Z}[t]$ is the N -th cyclotomic polynomial.

Fix a base scheme S over the ring $\mathbb{Z}[\zeta_N, (Nd_g)^{-1}]$. This means that S is a scheme such that Nd_g is invertible in $\Gamma(S, \mathcal{O}_S)$ and we have fixed a primitive N -th root of unity $\zeta_N \in \Gamma(S, \mathcal{O}_S)$. We can interpret ζ_N as an isomorphism of group schemes $\zeta_N : (\mathbb{Z}/N\mathbb{Z}) \xrightarrow{\cong} A[N]$. Let $V := \mathbb{Z}^{2g}$ and let $\phi_D : V \times V \rightarrow \mathbb{Z}$ be the symplectic pairing as defined before.

Definition 1.5. Let $g, N \geq 1$ be integers, $D = (d_1, d_2, \dots, d_g)$ be a tuple of positive integers with $d_1 | d_2 | \dots | d_g$ and $\gcd(d_g, N) = 1$ and S be a scheme over $\mathbb{Z}[\zeta_N, (Nd_g)^{-1}]$. Let A/S be an abelian scheme of relative dimension g over S and

$$\lambda : A/S \rightarrow \text{Pic}^0(A/S)$$

be a polarisation of type $D = (d_1, d_2, \dots, d_g)$. A *(symplectic) level N structure of A/S* is an isomorphism of group schemes

$$\eta : (V/NV) \xrightarrow{\cong} A[N]$$

such that the diagram

$$\begin{array}{ccc} (V/NV) \times (V/NV) & \xrightarrow{\phi_D} & (\mathbb{Z}/N\mathbb{Z}) \\ \eta \times \eta \downarrow & & \downarrow \zeta_N \\ A[N] \times A[N] & \xrightarrow{e_N^D} & \mu_N \end{array}$$

is commutative.

We can now look at the moduli stacks of polarised abelian schemes with level N structures.

Definition 1.6. Let $g, N \geq 1$ be integers and $D = (d_1, d_2, \dots, d_g)$ be a tuple of positive integers with $d_1 | d_2 | \dots | d_g$ and $\gcd(d_g, N) = 1$. Let $(Sch/\mathbb{Z}[\zeta_N, (Nd_g)^{-1}])$ be the category of schemes over $\mathbb{Z}[\zeta_N, (Nd_g)^{-1}]$. Let $\mathcal{A}'_{D,[N]}$ be the category fibred in groupoids over $(Sch/\mathbb{Z}[\zeta_N, (Nd_g)^{-1}])$ defined by its groupoid of sections $\mathcal{A}'_{D,[N]}(S)$ as follows:

For a scheme S over $\text{Spec}(\mathbb{Z}[\zeta_N, (Nd_g)^{-1}])$, let $\mathcal{A}'_{D,[N]}(S)$ be the groupoid whose objects are the triples $(A/S, \lambda, \eta)$, where A/S is an abelian scheme over S ,

$$\lambda : A/S \rightarrow \text{Pic}^0(A/S)$$

a polarisation of type D and

$$\eta : (V/NV) \xrightarrow{\cong} A[N]$$

a level N structure. The morphisms are S -isomorphisms of abelian schemes compatible with polarisations of type D and level N structures, i.e. a morphism between $(A/S, \lambda, \eta)$ and $(A'/S, \lambda', \eta')$ is a homomorphism of abelian schemes $\varphi : A/S \rightarrow A'/S$ such that $\varphi^*(\lambda') = \lambda$ and $\eta' = \varphi[N] \circ \eta$, where $\varphi[N]$ is the induced morphism on the group schemes of N -torsion points.

If we give the category $(Sch/\mathbb{Z}[\zeta_N, (Nd_g)^{-1}])$ the étale topology, then $\mathcal{A}'_{D,[N]}$ is a stack over the big étale site $(Sch/\mathbb{Z}[\zeta_N, (Nd_g)^{-1}])_{et}$.

The *moduli stack $\mathcal{A}_{D,[N]}$ of abelian schemes with polarisations of type D and level N structures* is defined as the stack $\mathcal{A}'_{D,[N]}$ viewed as a stack over the big étale site $(Sch/\mathbb{Z}[(Nd_g)^{-1}])_{et}$ with structure morphism given by the natural composition

$$\mathcal{A}_{D,[N]} := \left(\mathcal{A}'_{D,[N]} \rightarrow \text{Spec}(\mathbb{Z}[\zeta_N, (Nd_g)^{-1}]) \rightarrow \text{Spec}(\mathbb{Z}[(Nd_g)^{-1}]) \right).$$

If $d_g = 1$, then $\mathcal{A}_{D,[N]}$ is the *moduli stack $\mathcal{A}_{g,[N]}$ of principally polarised abelian schemes of relative dimension g with level N structures* over $(Sch/\mathbb{Z}[N^{-1}])_{et}$.

The moduli stacks of polarised schemes with level structures behave very well. In fact we have the following (cf. [MO], [FC]).

Theorem 1.7. Let $g, N \geq 1$ be integers and $D = (d_1, d_2, \dots, d_g)$ be a tuple of positive integers with $d_1 | d_2 | \dots | d_g$ and $\gcd(d_g, N) = 1$. The moduli stack $\mathcal{A}'_{D,[N]}$ is a quasi-projective smooth Deligne-Mumford

stack of finite type over $\mathrm{Spec}(\mathbb{Z}[\zeta_N, (Nd_g)^{-1}])$ with irreducible geometric fibers. Furthermore, if N is large enough with respect to D , then $\mathcal{A}'_{D,[N]}$ is a smooth quasi-projective scheme over $\mathrm{Spec}(\mathbb{Z}[\zeta_N, (Nd_g)^{-1}])$. The moduli stack $\mathcal{A}_{D,[N]}$ is a quasi-projective smooth Deligne-Mumford stack of finite type over $\mathrm{Spec}(\mathbb{Z}[(Nd_g)^{-1}])$ whose geometric fibers have $\varphi(N)$ irreducible components, where φ denotes Euler's function. Furthermore, if N is large enough with respect to D , then $\mathcal{A}_{D,[N]}$ is a smooth quasi-projective scheme over $\mathrm{Spec}(\mathbb{Z}[(Nd_g)^{-1}])$.

In particular for $d_g = 1$, the moduli stack $\mathcal{A}_{g,[N]}$ is a quasi-projective smooth Deligne-Mumford stack of finite type over $\mathrm{Spec}(\mathbb{Z}[N^{-1}])$. If in addition $N \geq 3$, then $\mathcal{A}_{g,[N]}$ is a smooth quasi-projective scheme over $\mathrm{Spec}(\mathbb{Z}[N^{-1}])$.

Proof. That $\mathcal{A}_{D,[N]}$ is a Deligne-Mumford stack over $\mathrm{Spec}(\mathbb{Z}[(Nd_g)^{-1}])$ follows along the same arguments as for \mathcal{A}_D in Theorem 1.4.

In the special case that $d_g = 1$, Serre's lemma [S2] shows that if $N \geq 3$, then the algebraic stack $\mathcal{A}_{g,[N]}$ is a smooth algebraic space over $\mathrm{Spec}(\mathbb{Z}[N^{-1}])$. Mumford proved using GIT methods [MFK] that $\mathcal{A}_{g,[N]}$ is in fact a smooth quasi-projective scheme of finite type over $\mathrm{Spec}(\mathbb{Z}[N^{-1}])$. These methods can also be adapted to the general case if N is large enough with respect to d_g (cf. [GeN, Theorem 2.3.1]). \square

Again in the following sections we will be mainly interested in the restriction of these moduli stacks $\mathcal{A}_{D,[N]}$ and $\mathcal{A}_{g,[N]}$ to the category of schemes over the fields \mathbb{Q} and $\bar{\mathbb{Q}}$.

Finally, we like to compare the algebraic with the complex analytic setting, which is easier to describe. So let us denote by $\mathcal{A}_{D,[N]}^{an}$ the complex analytification or the uniformisation of the moduli stack $\mathcal{A}_{D,[N]}$. It is again a complex analytic Deligne-Mumford stack over the big site $(AnSp)_{cl}$.

In fact, the moduli stack $\mathcal{A}_{D,[N]}^{an}$ is again given as a quotient stack. Let us explain this. Consider a complex abelian variety of dimension g , $A = W/\Lambda$, where W is a complex vector space of dimension g and Λ is a lattice. The set of N -torsion points $A[N]$ can be identified with the finite group $N^{-1}\Lambda/\Lambda \cong (\mathbb{Z}/N\mathbb{Z})^{2g}$. Assume that E is a polarisation of type $D = (d_1, \dots, d_g)$ with d_g and N coprime. The alternating form $E : \Lambda \times \Lambda \rightarrow \mathbb{Z}$ can be extended to a non-degenerate symplectic form on $\Lambda \otimes \mathbb{Q}$. The Weil pairing $(\alpha, \beta) \mapsto \exp(2\pi i N E(\alpha, \beta))$ defines a symplectic non-degenerate form

$$e_N^D : A[N] \times A[N] \rightarrow \mu_N,$$

and if we choose a primitive N -th root of unity, we can think of the Weil pairing again as a symplectic non-degenerate form with values in $\mathbb{Z}/N\mathbb{Z}$. Consider the subgroup $\Gamma_D(N)$ of $\mathrm{Sp}_D(\mathbb{Z})$ consisting of the automorphisms of the pair (Λ, E) , which induce the trivial action on $\Lambda/N\Lambda$. One can prove that there is a natural bijection between the set of isomorphism classes of polarised abelian varieties of type D equipped with a level N structure and the quotient $\mathfrak{H}_g/\Gamma_D(N)$. The group $\Gamma_D(N)$ acts properly discontinuously on \mathfrak{H}_g , so this quotient has the structure of a complex analytic orbifold, hence the moduli stack $\mathcal{A}_{D,[N]}^{an}$ can be described as the quotient stack $\mathcal{A}_{D,[N]}^{an} = [\mathfrak{H}_g/\Gamma_D(N)]$.

If N is large enough with respect to the type $D = (d_1, d_2, \dots, d_g)$ the group $\mathrm{Sp}_D(\mathbb{Z})$ does not contain torsion elements and acts freely on the Siegel upper half-space \mathfrak{H}_g and we have that

$$\mathcal{A}_{D,[N]}^{an} = \mathfrak{H}_g/\Gamma_D(N)$$

as a quotient being a smooth complex analytic space (cf. [GeN]).

Especially again in the case of principally polarised complex abelian varieties, i.e. when $d_g = 1$, we get that

$$\mathcal{A}_{g,[N]}^{an} = [\mathfrak{H}_g/\Gamma_{2g}(N)],$$

where the group $\Gamma_{2g}(N)$ is the subgroup of $\mathrm{Sp}(2g, \mathbb{Z})$ corresponding to the symplectic matrices which are congruent to the identity modulo N . And if $N \geq 3$, then we get the quotient

$$\mathcal{A}_{g,[N]}^{an} = \mathfrak{H}_g/\Gamma_{2g}(N),$$

which is a smooth complex analytic space again.

2. ÉTALE AND COMPLEX ANALYTIC HOMOTOPY TYPES FOR DELIGNE-MUMFORD STACKS

In this section we briefly recall the constructions of étale and complex analytic homotopy types for Deligne-Mumford stacks following the presentation in [FN, Section 2], which is based on [AM], [C] and [Mo].

Let \mathfrak{E} be a topos with an initial object \emptyset . An object X of \mathfrak{E} is connected, if whenever $X = X_1 \coprod X_2$, either X_1 or X_2 is the initial object \emptyset . A topos \mathfrak{E} is locally connected if every object of \mathfrak{E} is a coproduct of connected objects. For a locally connected topos \mathfrak{E} define the connected component functor π as the functor

$$\pi : \mathfrak{E} \rightarrow (\mathbf{Sets})$$

associating to any object X of \mathfrak{E} its set $\pi(X)$ of connected components.

Let Δ be the category of simplices, i.e. the category whose objects are sets $[n] = \{0, 1, 2, \dots, n\}$ and whose morphisms are non-decreasing maps. We denote by Δ_n the full subcategory of Δ of all sets $[k]$ with $k \leq n$. A simplicial object in a topos \mathfrak{E} is a functor $X_\bullet : \Delta^{op} \rightarrow \mathfrak{E}$ denoted by $X_\bullet = \{X_n\}_{n \geq 0}$. We will write $\Delta^{op}\mathfrak{E}$ for the category of all simplicial objects in \mathfrak{E} . The restriction or n -truncation functor

$$(-)^{(n)} : \Delta^{op}\mathfrak{E} \rightarrow \Delta_n^{op}\mathfrak{E}$$

has left and right adjoint functors, the n -th skeleton and n -th coskeleton functor

$$\text{sk}_n, \text{cosk}_n : \Delta_n^{op}\mathfrak{E} \rightarrow \Delta^{op}\mathfrak{E}.$$

Definition 2.1. A *hypercovering* of a topos \mathfrak{E} is a simplicial object $U_\bullet = \{U_n\}_{n \geq 0}$ such that the morphisms

$$U_0 \rightarrow *$$

$$U_{n+1} \rightarrow \text{cosk}_n(U_\bullet)_{n+1}$$

are epimorphisms, where $*$ is the final object of \mathfrak{E} .

If S is a set and X an object of a topos \mathfrak{E} define $S \otimes X := \coprod_{s \in S} X$. If S_\bullet is a simplicial set and X_\bullet a simplicial object of \mathfrak{E} , define

$$S_\bullet \otimes X_\bullet : \Delta^{op} \rightarrow \mathfrak{E}$$

to be the simplicial object given by $(S_\bullet \otimes X_\bullet)_n := S_n \otimes X_n$.

Let $\Delta[m] = \text{Hom}_\Delta(-, [m])$ be the standard m -simplicial set, i.e. the functor $\Delta[m] : \Delta^{op} \rightarrow (\text{Sets})$ represented by the set $[m]$.

Two morphisms $f, g : X_\bullet \rightarrow Y_\bullet$ are called strictly homotopic if there is a morphism $H : X_\bullet \otimes \Delta[1] \rightarrow Y_\bullet$, called strict homotopy, such that the following diagram is commutative

$$\begin{array}{ccc} X_\bullet = X_\bullet \otimes \Delta[0] & & \\ \text{id} \otimes d^0 \downarrow & \searrow f & \\ X_\bullet \otimes \Delta[1] & \xrightarrow{H} & Y_\bullet \\ \text{id} \otimes d^1 \uparrow & \nearrow g & \\ X_\bullet = X_\bullet \otimes \Delta[0] & & \end{array}$$

Two morphisms $f, g : X_\bullet \rightarrow Y_\bullet$ are called homotopic, if they can be related by a chain of strict homotopies. Homotopy is the equivalence relation generated by strict homotopy.

Let $\mathcal{H} = \mathcal{H}(\Delta^{op}(\text{Sets}))$ be the homotopy category of simplicial sets and $\mathcal{HR}(\mathfrak{E})$ be the homotopy category of hypercoverings of \mathfrak{E} , i.e., the category of hypercoverings of \mathfrak{E} with their morphisms up to homotopy.

It can be shown that the opposite category $\mathcal{HR}(\mathfrak{E})^{op}$ is in fact a filtering category [AM].

Let $\text{pro} - \mathcal{H}$ denote the category of pro-objects in the category \mathcal{H} , i.e. the category of (contravariant) functors $X : \mathcal{I} \rightarrow \mathcal{H}$ from a filtering index category \mathcal{I} to \mathcal{H} . We will write normally $X = \{X_i\}_{i \in \mathcal{I}}$ to indicate that we think of pro-objects X as inverse systems of objects of \mathcal{H} (cf. [AM, Appendix]).

The connected component functor $\pi : \mathfrak{E} \rightarrow (\text{Sets})$ induces a functor

$$\Delta^{op} \pi : \Delta^{op} \mathfrak{E} \rightarrow \Delta^{op}(\text{Sets}).$$

If we pass to homotopy categories and restrict to hypercoverings of \mathfrak{E} we obtain a functor

$$\pi : \mathcal{HR}(\mathfrak{E}) \rightarrow \text{pro} - \mathcal{H}.$$

Finally we can define the Artin-Mazur homotopy type of a locally connected topos, following [AM, §9], [Mo].

Definition 2.2. Let \mathfrak{E} be a locally connected topos. The *Artin-Mazur homotopy type* of \mathfrak{E} is the pro-object

$$\{\mathfrak{E}\} = \{\pi(U_\bullet)\}_{U_\bullet \in \mathcal{HR}(\mathfrak{E})}.$$

in the homotopy category of simplicial sets. The Artin-Mazur homotopy type is functorial with respect to morphisms of topoi, the associated functor $\{-\}$ is called the *Verdier functor*.

If \mathfrak{E} is a locally connected topos and x a point of the topos \mathfrak{E} , i.e. a morphism of topoi $x : (\text{Sets}) \rightarrow \mathfrak{E}$, we can also define the homotopy pro-groups $\pi_n(\mathfrak{E}, x)$ of \mathfrak{E} for $n \geq 1$. (cf. [AM], [Mo] and [Z]).

Now we will recall the definition of the étale homotopy type for the topos of sheaves on the small étale site of an algebraic Deligne-Mumford stack (cf. [FN, Section 2.2]). For the general theory of algebraic stacks we refer again to [LMB] and [Stacks].

Let \mathcal{S} be an algebraic Deligne-Mumford stack over the big étale site of the category of schemes (Sch/B) over a base scheme B .

Then we can consider its small étale site \mathcal{S}_{et} . The objects are étale 1-morphisms $x : X \rightarrow \mathcal{S}$, where X is a scheme, morphisms are morphisms over \mathcal{S} , i.e., commutative diagrams of 1-morphisms of the form

$$\begin{array}{ccc} X & \xrightarrow{\quad} & Y \\ & \searrow & \swarrow \\ & \mathcal{S} & \end{array}$$

and the coverings are the étale coverings of the schemes. We now let $\mathfrak{E}_{et} = \mathfrak{Sh}(\mathcal{S}_{et})$ be the associated étale topos, i.e. the category of sheaves on the small étale site \mathcal{S}_{et} of the algebraic stack \mathcal{S} . It turns out that this topos \mathfrak{E}_{et} is in fact a locally connected topos (cf. [Z, 3.1]).

We define the étale homotopy type of an algebraic Deligne-Mumford stack as the Artin-Mazur homotopy type of its étale topos (cf. [FN]).

Definition 2.3. Let \mathcal{S} be an algebraic Deligne-Mumford stack. The *étale homotopy type* $\{\mathcal{S}\}_{et}$ of \mathcal{S} is the pro-object in the homotopy category of simplicial sets given as

$$\{\mathcal{S}\}_{et} = \{\mathfrak{E}_{et}\}.$$

The *étale homotopy groups* of the stack \mathcal{S} are defined as the Artin-Mazur homotopy pro-groups $\pi_n^{et}(\mathcal{S}, x) = \pi_n(\mathfrak{E}_{et}, x)$ of the topos \mathfrak{E}_{et} .

For $n = 1$ this gives the étale fundamental group $\pi_1^{et}(\mathcal{S}, x)$ of the algebraic stack \mathcal{S} , which is discussed in detail also in Noohi [No] and Zoonekynd [Z].

As was shown in [FN, Section 2.2] we can determine the étale homotopy type of a Deligne-Mumford stack \mathcal{S} directly from that of a hypercovering X_\bullet , i.e. from a simplicial scheme over \mathcal{S} which can often be constructed directly for example from an étale atlas of the Deligne-Mumford stack. This is done using the following generalisation of the homotopy descent theorem for simplicial schemes of Cox [C, Theorem IV.2] (cf. also [O, Theorem 3]). The étale homotopy type $\{X_\bullet\}_{et}$ of the simplicial scheme X_\bullet is defined as in Cox [C, Chap. III]. We have the following homotopy descent theorem [FN, Theorem 2.3].

Theorem 2.4. If \mathcal{S} is an algebraic Deligne-Mumford stack with small étale site \mathcal{S}_{et} and X_\bullet a simplicial scheme which is a hypercovering of the étale topos \mathfrak{E}_{et} , then there is a weak homotopy equivalence of pro-simplicial sets

$$\{\mathcal{S}\}_{et} \simeq \{X_\bullet\}_{et}.$$

where $\{X_\bullet\}_{et}$ is the étale homotopy type of the simplicial scheme X_\bullet .

Similarly we can consider Artin-Mazur homotopy types of analytic Deligne-Mumford stacks in the context of complex analytic spaces (cf. [FN, Section 2.3]). We also refer to [T, Chap. 5] for the general theory and properties of complex analytic stacks and analytifications of algebraic stacks.

Let \mathcal{T} be an analytic Deligne-Mumford stack over the big site of local isomorphisms of the category of complex analytic spaces (AnSp/C) over a base complex analytic space C .

For an analytic Deligne-Mumford stack \mathcal{T} we can define its small site of local isomorphisms \mathcal{T}_{cl} given by local isomorphisms $x : X \rightarrow \mathcal{T}$, where X is a complex analytic space and morphisms are morphisms of complex analytic spaces over \mathcal{T} and the coverings are given by families of local isomorphisms (cf. [M1], and [C, Chap. IV, §3]). Let $\mathfrak{E}_{cl} = \mathfrak{Sh}(\mathcal{T}_{cl})$ be the associated topos of local isomorphisms, i.e. the category of sheaves on the small site of local isomorphisms of the analytic stack \mathcal{T} . It is again a locally connected topos and we can define the complex analytic homotopy type of \mathcal{T} .

Definition 2.5. Let \mathcal{T} be an analytic Deligne-Mumford stack. The *complex analytic* or *classical homotopy type* $\{\mathcal{T}\}_{cl}$ of \mathcal{T} is the pro-object in the homotopy category of simplicial sets given as

$$\{\mathcal{T}\}_{cl} = \{\mathfrak{E}_{cl}\}.$$

The *classical homotopy groups* of the stack \mathcal{T} are defined as the Artin-Mazur homotopy pro-groups $\pi_n^{cl}(\mathcal{T}, x) = \pi_n(\mathfrak{E}_{cl}, x)$ of the topos \mathfrak{E}_{cl} .

In the complex analytic setting we have a very explicit description of the classical homotopy type $\{\mathcal{T}\}_{cl}$ of an analytic Deligne-Mumford stack using hypercoverings and underlying topological spaces. Namely, if \mathcal{T} is an analytic Deligne-Mumford stack and X_\bullet is a simplicial analytic space, which is also a hypercovering of the topos \mathfrak{E}_{cl} , then homotopy descent gives again a weak homotopy equivalence

$$\{\mathcal{T}\}_{cl} \simeq \{X_\bullet\}_{cl}$$

where the classical homotopy type $\{X_\bullet\}_{cl}$ of the small site of local isomorphisms of the hypercovering X_\bullet is given as $\{X_\bullet\}_{cl} = \Delta \text{Sin}(X_\bullet)$. Here $\text{Sin}(X_\bullet)$ is the bisimplicial set given in bidegree s, t by $\text{Sin}_t(X_s)$ and Δ is the diagonal functor (cf. [C, Chap. IV, §3] and [F, Chap. 8]). There is a canonical weak homotopy equivalence

$$\Delta \text{Sin}(X_\bullet) \xrightarrow{\simeq} \text{Sin}(|X_\bullet|)$$

where $|X_\bullet|$ denotes the geometric realization of the simplicial space X_\bullet . From this we get a weak homotopy equivalence

$$\{\mathcal{T}\}_{cl} \simeq \text{Sin}(|X_\bullet|).$$

So the classical homotopy type of an analytic Deligne-Mumford stack is determined by a complex analytic hypercovering. In [O] it is shown that the construction is independent of the actual choice of the hypercovering X_\bullet .

Now we fix an embedding $\bar{\mathbb{Q}} \hookrightarrow \mathbb{C}$ of the algebraic closure of the rational numbers in the complex numbers. For a given algebraic Deligne-Mumford stack \mathcal{S} over $\mathrm{Spec}(\bar{\mathbb{Q}})$ let \mathcal{S}^{an} be the associated analytic Deligne-Mumford stack (cf. [T, Chap. 5]). Similarly for any scheme X over $\mathrm{Spec}(\bar{\mathbb{Q}})$, let X^{an} denote the complex analytic space associated with the \mathbb{C} -valued points $X(\mathbb{C})$ of the scheme X .

Both the étale and the classical homotopy type are determined by a hypercovering of their respective topoi, and we recall the following general comparison theorem between étale and classical homotopy types of Deligne-Mumford stacks [FN, Theorem 2.4], which is a consequence of the comparison theorem for homotopy types of simplicial schemes by Cox [C, Theorem IV.8] and Friedlander [F, Theorem 8.4].

Theorem 2.6. Let \mathcal{S} be an algebraic Deligne-Mumford stack over $\bar{\mathbb{Q}}$ and X_\bullet a simplicial scheme which is of finite type over $\bar{\mathbb{Q}}$ which is a hypercovering of the topos $\mathfrak{E}_{et} = \mathfrak{Sh}(\mathcal{S}_{et})$. If X_\bullet^{an} is the associated simplicial complex analytic space of X_\bullet , then there is a weak homotopy equivalence of pro-simplicial sets

$$\{\mathcal{S}\}_{et}^\wedge \simeq \mathrm{Sin}(|X_\bullet^{an}|)^\wedge.$$

where $^\wedge$ denotes the Artin-Mazur profinite completion functor on the homotopy category of simplicial sets.

This comparison theorem now allows to determine the étale homotopy types over $\bar{\mathbb{Q}}$ of the moduli stacks of polarised abelian schemes we introduced in the last section.

3. ÉTALE HOMOTOPY TYPES OF MODULI STACKS OF POLARISED ABELIAN SCHEMES

We will now prove our main theorem on the étale homotopy type of the moduli stacks of polarised abelian schemes using the general comparison theorem for étale homotopy types as outlined in the last section.

Let $g \geq 1$ be a positive integer and $D = (d_1, d_2, \dots, d_g)$ be a tuple of integers with $d_1 | d_2 | \dots | d_g$. We want to study the étale homotopy type of the moduli stack \mathcal{A}_D of abelian schemes with polarisations of type D .

We denote by $\mathcal{A}_D \otimes \bar{\mathbb{Q}}$ the restriction of the moduli stack \mathcal{A}_D to the category of schemes over $\bar{\mathbb{Q}}$, i.e. the extension of scalars of \mathcal{A}_D to the algebraic closure $\bar{\mathbb{Q}}$ of the field \mathbb{Q} of rational numbers.

We also fix an embedding $\bar{\mathbb{Q}} \hookrightarrow \mathbb{C}$ of the algebraic closure of the rationals in the complex numbers. The analytic stack \mathcal{A}_D^{an} is precisely

the complex analytification $(\mathcal{A}_D \otimes \bar{\mathbb{Q}})^{an}$ of the Deligne-Mumford stack $\mathcal{A}_D \otimes \bar{\mathbb{Q}}$ and we can now determine the étale homotopy type of $\mathcal{A}_D \otimes \bar{\mathbb{Q}}$.

First we determine the classical homotopy type of the complex analytification \mathcal{A}_D^{an} .

Theorem 3.1. Let $g \geq 1$ be a positive integer and $D = (d_1, d_2, \dots, d_g)$ be a tuple of integers with $d_1 | d_2 | \dots | d_g$. There is a weak homotopy equivalence of simplicial sets

$$\{\mathcal{A}_D^{an}\}_{cl} \simeq \text{Sin}(|\text{BSp}_D(\mathbb{Z})|).$$

In particular for principal polarisations we have

$$\{\mathcal{A}_g^{an}\}_{cl} \simeq \text{Sin}(|\text{BSp}(2g, \mathbb{Z})|).$$

Proof. Since the moduli stack \mathcal{A}_D^{an} is given as the quotient stack

$$\mathcal{A}_g^{an} = [\mathfrak{H}_g / \text{Sp}_D(\mathbb{Z})],$$

we have the following cartesian diagram

$$\begin{array}{ccc} \text{Isom}(\pi, \pi) = \mathfrak{H}_g \times_{\mathcal{A}_D^{an}} \mathfrak{H}_g & \longrightarrow & \mathfrak{H}_g \\ \downarrow & & \downarrow \pi \\ \mathfrak{H}_g & \xrightarrow{\pi} & \mathcal{A}_D^{an} \end{array}$$

where the morphism π is an atlas, which is a local isomorphism and both projections from $\text{Isom}(\pi, \pi) = \mathfrak{H}_g \times_{\mathcal{A}_D^{an}} \mathfrak{H}_g$ are local isomorphisms. The atlas $\pi : \mathfrak{H}_g \rightarrow \mathcal{A}_D^{an}$ corresponds to the trivial principal $\text{Sp}_D(\mathbb{Z})$ -bundle $\mathfrak{H}_g \times \text{Sp}_D(\mathbb{Z})$ together with the $\text{Sp}_D(\mathbb{Z})$ -equivariant morphism given by the right action $\rho : \text{Sp}_D(\mathbb{Z}) \times \mathfrak{H}_g \rightarrow \mathfrak{H}_g$ (cf. [DM], [Go]). Therefore we have

$$\text{Isom}(\pi, \pi) = \mathfrak{H}_g \times_{\mathcal{A}_D^{an}} \mathfrak{H}_g \cong \mathfrak{H}_g \times \text{Sp}_D(\mathbb{Z}).$$

By induction on the number of factors, we get an isomorphism

$$\mathfrak{H}_g \times_{\mathcal{A}_D^{an}} \mathfrak{H}_g \dots \times_{\mathcal{A}_D^{an}} \mathfrak{H}_g \cong \mathfrak{H}_g \times \text{Sp}_D(\mathbb{Z}) \times \dots \times \text{Sp}_D(\mathbb{Z}).$$

This allows us to determine the classical homotopy type of the uniformization \mathcal{A}_D^{an} . Let $\text{cosk}_0^{\mathcal{A}_D^{an}}(\mathfrak{H}_g)_\bullet$ be the relative Čech nerve. It is a hypercovering associated to the atlas $\pi : \mathfrak{H}_g \rightarrow \mathcal{A}_D^{an}$. The induction above shows that, the m -simplex $\text{cosk}_0^{\mathcal{A}_D^{an}}(\mathfrak{H}_g)_m$ of the Čech nerve $\text{cosk}_0^{\mathcal{A}_D^{an}}(\mathfrak{H}_g)_\bullet$ is given by the $(m+1)$ -tuple fiber product of copies of \mathfrak{H}_g over \mathcal{A}_D^{an} , so we have:

$$\text{cosk}_0^{\mathcal{A}_D^{an}}(\mathfrak{H}_g)_m \cong \mathfrak{H}_g \times \text{Sp}_D(\mathbb{Z}) \times \text{Sp}_D(\mathbb{Z}) \times \dots \times \text{Sp}_D(\mathbb{Z}).$$

And by definition, we get

$$\text{cosk}_0^{\mathcal{A}_D^{an}}(\mathfrak{H}_g)_\bullet \cong \mathfrak{H}_g \times \mathcal{N}(\text{Sp}_D(\mathbb{Z}))_\bullet.$$

where the simplicial set $\mathcal{N}(\mathrm{Sp}_D(\mathbb{Z}))_\bullet$ is the simplicial nerve of the group $\mathrm{Sp}_D(\mathbb{Z})$, viewed as a category with one object and with $\mathrm{Sp}_D(\mathbb{Z})$ as the set of morphisms. The geometric realization of $\mathcal{N}(\mathrm{Sp}_D(\mathbb{Z}))_\bullet$ is the classifying space $\mathrm{BSp}_D(\mathbb{Z})$. Therefore after geometric realization we have a homeomorphism of topological spaces

$$|\mathrm{cosk}_0^{\mathcal{A}_D^{an}}(\mathfrak{H}_g)_\bullet| \cong \mathfrak{H}_g \times \mathrm{BSp}_D(\mathbb{Z}).$$

The Siegel upper half space $\mathfrak{H}_g = \mathrm{Sp}(2g, \mathbb{R})/U(g)$ is a homogeneous space given as the quotient $\mathrm{Sp}(2g, \mathbb{R})/U(g)$, where the unitary group $U(g)$ is a maximal compact subgroup of $\mathrm{Sp}(2g, \mathbb{R})$. Moreover one can show that \mathfrak{H}_g is contractible (cf. [McDS, 2.20]). Therefore we finally conclude that there is a homotopy equivalence

$$|\mathrm{cosk}_0^{\mathcal{A}_D^{an}}(\mathfrak{H}_g)_\bullet| \simeq \mathrm{BSp}_D(\mathbb{Z})$$

which determines the classical homotopy type of the uniformisation \mathcal{A}_D^{an} of the moduli stack \mathcal{A}_D .

In the case that the polarisations are principal, i.e. $d_g = 1$ we have especially

$$|\mathrm{cosk}_0^{\mathcal{A}_g^{an}}(\mathfrak{H}_g)_\bullet| \simeq \mathrm{BSp}(2g, \mathbb{Z})$$

determining the classical homotopy of the uniformisation \mathcal{A}_g^{an} of the moduli stack \mathcal{A}_g of principally polarised abelian schemes. \square

Now we can determine the étale homotopy types of the moduli stacks \mathcal{A}_D using the comparison theorem (cf. Theorem 2.6).

Theorem 3.2. Let $g \geq 1$ be a positive integer and $D = (d_1, d_2, \dots, d_g)$ be a tuple of integers with $d_1|d_2|\dots|d_g$. There is a weak homotopy equivalence of pro-simplicial sets

$$\{\mathcal{A}_D \otimes \bar{\mathbb{Q}}\}_{et}^\wedge \simeq K(\mathrm{Sp}_D(\mathbb{Z}), 1)^\wedge.$$

where $^\wedge$ denotes Artin-Mazur profinite completion. In particular for principal polarisations we have

$$\{\mathcal{A}_g \otimes \bar{\mathbb{Q}}\}_{et}^\wedge \simeq K(\mathrm{Sp}(2g, \mathbb{Z}), 1)^\wedge.$$

Proof. The moduli stack $\mathcal{A}_g \otimes \bar{\mathbb{Q}}$ is a Deligne-Mumford stack, so there exists a scheme X over $\bar{\mathbb{Q}}$ and an étale surjective morphism $x : X \rightarrow \mathcal{A}_g \otimes \bar{\mathbb{Q}}$. The relative Čech nerve $\mathrm{cosk}_0^{\mathcal{A}_g \otimes \bar{\mathbb{Q}}}(X)_\bullet$ for this morphism defines a hypercovering of the stack $\mathcal{A}_g \otimes \bar{\mathbb{Q}}$. Similarly, $\mathrm{cosk}_0^{(\mathcal{A}_g \otimes \bar{\mathbb{Q}})^{an}}(X^{an})_\bullet$ is a hypercovering of the complex analytic stack \mathcal{A}_g^{an} . Here X^{an} denotes the associated complex analytic space of the covering scheme X over $\bar{\mathbb{Q}}$.

The homotopy descent theorem (cf. Theorem 2.4) and rigidity of étale homotopy types with respect to base change of algebraically closed

fields in characteristic 0 (cf. [SGA1, Exp. X, Cor. 1.8] and [AM]) implies that there are weak equivalences of pro-simplicial sets:

$$\{\mathcal{A}_D \otimes \bar{\mathbb{Q}}\}_{et} \simeq \{\mathrm{cosk}_0^{\mathcal{A}_D \otimes \bar{\mathbb{Q}}}(X)_\bullet\}_{et}.$$

The étale homotopy type does not change under base change for algebraically closed fields [AM, Cor. 12.12] and we have:

$$\{\mathrm{cosk}_0^{\mathcal{A}_D \otimes \bar{\mathbb{Q}}}(X)_\bullet\}_{et}^\wedge \simeq \{\mathrm{cosk}_0^{\mathcal{A}_D \otimes \bar{\mathbb{Q}}}(X)_\bullet \otimes_{\bar{\mathbb{Q}}} \mathbb{C}\}_{et}^\wedge.$$

and the comparison theorem for simplicial schemes (cf. Cox [C, Thm. IV.8]) now shows that there is also a weak equivalence after profinite completions:

$$\{\mathrm{cosk}_0^{\mathcal{A}_D \otimes \bar{\mathbb{Q}}}(X)_\bullet \otimes_{\bar{\mathbb{Q}}} \mathbb{C}\}_{et}^\wedge \simeq \mathrm{Sin}(|\mathrm{cosk}_0^{(\mathcal{A}_D \otimes \bar{\mathbb{Q}})^{an}}(X^{an})_\bullet|)^\wedge.$$

Because $\mathrm{cosk}_0^{(\mathcal{A}_D \otimes \bar{\mathbb{Q}})^{an}}(X^{an})_\bullet$ is a hypercovering of the complex analytic stack \mathcal{A}_D^{an} we have a weak equivalence:

$$\{\mathcal{A}_D^{an}\}_{cl} \simeq \mathrm{Sin}(|\mathrm{cosk}_0^{(\mathcal{A}_D \otimes \bar{\mathbb{Q}})^{an}}(X^{an})_\bullet|)$$

From the determination of the classical homotopy type of \mathcal{A}_D^{an} in Theorem 3.1 we get therefore finally the following weak homotopy equivalences:

$$\{\mathcal{A}_D \otimes \bar{\mathbb{Q}}\}_{et}^\wedge \simeq \{\mathcal{A}_D^{an}\}_{cl}^\wedge \simeq \mathrm{BSp}_D(\mathbb{Z})^\wedge = K(\mathrm{Sp}_D(\mathbb{Z}), 1)^\wedge.$$

The particular case for the moduli stack \mathcal{A}_g follows now immediately from this as $d_g = 1$. \square

As an immediate consequence we also get the étale homotopy type of the moduli stack \mathcal{M}_{ell} of elliptic curves [DR], which is simply given as the moduli stack \mathcal{A}_1 of principally polarised abelian schemes. In this case we have the complex analytic orbifold $\mathcal{M}_{ell}^{an} = [\mathfrak{H}_2/SL(2, \mathbb{Z})]$ and so with the same arguments as before we get

Corollary 3.3. There is a weak homotopy equivalence of pro-simplicial sets

$$\{\mathcal{M}_{ell} \otimes \bar{\mathbb{Q}}\}_{et}^\wedge \simeq K(\mathrm{SL}(2, \mathbb{Z}), 1)^\wedge.$$

where $\mathrm{SL}(2, \mathbb{Z})$ is the integral special linear group and $^\wedge$ denotes Artin-Mazur profinite completion.

We can interpret the main result above also in terms of étale homotopy groups and we have the following corollary:

Corollary 3.4. Let x be any point in the stack $\mathcal{A}_D \otimes \bar{\mathbb{Q}}$, then we have for the étale fundamental group:

$$\pi_1^{et}(\mathcal{A}_D \otimes \bar{\mathbb{Q}}, x) \cong \mathrm{Sp}_D(\mathbb{Z})^\wedge$$

and the higher étale homotopy pro-groups are trivial, i.e. for all $n > 1$ we have:

$$\pi_n^{et}(\mathcal{A}_D \otimes \bar{\mathbb{Q}}, x) = \{0\}.$$

Especially for principally polarised schemes we have

$$\pi_1^{et}(\mathcal{A}_g \otimes \bar{\mathbb{Q}}, x) = \prod_{p \text{ prime}} \mathrm{Sp}(2g, \mathbb{Z}_p).$$

Proof. By Theorem 3.2 and by [AM, 3.7], we conclude that

$$\pi_1^{et}(\mathcal{A}_D \otimes \bar{\mathbb{Q}}, x) \cong \mathrm{Sp}_D(\mathbb{Z})^\wedge.$$

and that the higher étale homotopy pro-groups $\pi_n^{et}(\mathcal{A}_D \otimes \bar{\mathbb{Q}}, x)$ are trivial for all $n > 1$.

The affirmative solution of the congruence subgroup problem for the discrete group $\mathrm{Sp}(2g, \mathbb{Z})$ (cf. [BMS]) in addition implies that

$$\mathrm{Sp}(2g, \mathbb{Z})^\wedge = \mathrm{Sp}(2g, \widehat{\mathbb{Z}}) \cong \prod_{p \text{ prime}} \mathrm{Sp}(2g, \mathbb{Z}_p).$$

and so we get the desired description of the étale fundamental group for the moduli stack \mathcal{A}_g of principally polarised abelian schemes. \square

The Grothendieck exact sequence of étale fundamental groups now allows to relate the étale fundamental groups of the moduli stacks $\mathcal{A}_g \otimes \mathbb{Q}$ with the profinite completions $\mathrm{Sp}(2g, \widehat{\mathbb{Z}})$ and with the absolute Galois group $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$.

Corollary 3.5. Let x be any point in the stack $\mathcal{A}_g \otimes \bar{\mathbb{Q}}$, then there is a short exact sequence of profinite groups

$$1 \rightarrow \prod_{p \text{ prime}} \mathrm{Sp}(2g, \mathbb{Z}_p) \rightarrow \pi_1^{et}(\mathcal{A}_g \otimes \bar{\mathbb{Q}}, x) \rightarrow \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}) \rightarrow 1.$$

Let us also remark that the group $\mathrm{Sp}(2g, \mathbb{Z})$ is not a good group in the sense of Serre (cf. [Mk, 3.16], [S1]), which implies, by the theorem of Artin-Mazur [AM, 6.6], that the profinite completion $K(\mathrm{Sp}(2g, \mathbb{Z}), 1)^\wedge$ is not weakly equivalent to $K(\mathrm{Sp}(2g, \widehat{\mathbb{Z}}), 1)$.

To finish this section we determine the étale homotopy types of the moduli stacks of polarised abelian schemes with level structures. So let again $g, N \geq 1$ be integers and $D = (d_1, d_2, \dots, d_g)$ be a tuple of positive integers with $d_1 | d_2 | \dots | d_g$ and additionally $\mathrm{gcd}(d_g, N) = 1$.

First, for the complex analytic stack $\mathcal{A}_{D, [N]}$ we get the following

Theorem 3.6. Let $g, N \geq 1$ be integers and $D = (d_1, d_2, \dots, d_g)$ be a tuple of positive integers with $d_1 | d_2 | \dots | d_g$ and $\gcd(d_g, N) = 1$. There is a weak homotopy equivalence of simplicial sets

$$\{\mathcal{A}_{D,[N]}^{an}\}_{cl} \simeq \text{Sin}(|\text{B}\Gamma_D(N)|).$$

For principal polarisations with level N structures we have in particular

$$\{\mathcal{A}_{g,[N]}^{an}\}_{cl} \simeq \text{Sin}(|\text{B}\Gamma_{2g}(N)|).$$

Proof. This follows as in the proof of Theorem 3.1 because the moduli stack $\mathcal{A}_{D,[N]}^{an}$ is given as the quotient stack $[\mathfrak{H}_g/\Gamma_D(N)]$ and in the case $d_g = 1$ of principal polarisations we have $\mathcal{A}_{g,[N]}^{an} = [\mathfrak{H}_g/\Gamma_{2g}(N)]$. \square

From this we can derive now the étale homotopy types of the moduli stacks $\mathcal{A}_{D,[N]}$ and $\mathcal{A}_{g,[N]}$.

Theorem 3.7. Let $g, N \geq 1$ be integers and $D = (d_1, d_2, \dots, d_g)$ be a tuple of positive integers with $d_1 | d_2 | \dots | d_g$ and $\gcd(d_g, N) = 1$. There is a weak homotopy equivalence of pro-simplicial sets

$$\{\mathcal{A}_{D,[N]} \otimes \bar{\mathbb{Q}}\}_{et}^\wedge \simeq K(\Gamma_D(N), 1)^\wedge.$$

where $^\wedge$ denotes Artin-Mazur profinite completion. In particular for principal polarisations with level N structures we have

$$\{\mathcal{A}_{g,[N]} \otimes \bar{\mathbb{Q}}\}_{et}^\wedge \simeq K(\Gamma_{2g}(N), 1)^\wedge.$$

Proof. The proof is the obvious variation of the proof of Theorem 3.2 using Theorem 3.6 and the comparison theorem for étale homotopy types of Deligne-Mumford stacks. \square

As a final consequence we state again the natural interpretation in terms of étale fundamental groups of algebraic stacks.

Corollary 3.8. Let x be any point in the stack $\mathcal{A}_{g,[N]} \otimes \bar{\mathbb{Q}}$, then we have for the étale fundamental group:

$$\pi_1^{et}(\mathcal{A}_{D,[N]} \otimes \bar{\mathbb{Q}}, x) \cong \Gamma_D(N)^\wedge$$

and the higher étale homotopy pro-groups are trivial, i.e. for all $n > 1$ we have:

$$\pi_n^{et}(\mathcal{A}_{D,[N]} \otimes \bar{\mathbb{Q}}, x) = \{0\}.$$

Especially for principally polarised schemes we have

$$\pi_1^{et}(\mathcal{A}_{g,[N]} \otimes \bar{\mathbb{Q}}, x) = \Gamma_{2g}(N)^\wedge.$$

Let us also remark that we have the following short exact sequence of étale fundamental groups.

Corollary 3.9. Let x be any point in the stack $\mathcal{A}_{g,[N]} \otimes \bar{\mathbb{Q}}$, then there is a short exact sequence of profinite groups

$$1 \rightarrow \Gamma_{2g}(N)^\wedge \rightarrow \pi_1^{et}(\mathcal{A}_{g,[N]} \otimes \bar{\mathbb{Q}}, x) \rightarrow \text{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}) \rightarrow 1.$$

Again it turns out that the groups $\Gamma_{2g}(N)$ are not good groups either and so the profinite completion $K(\Gamma_{2g}(N), 1)^\wedge$ is not weakly equivalent to $K(\Gamma_{2g}(N)^\wedge, 1)$.

4. THE TORELLI MORPHISM AND ÉTALE HOMOTOPY TYPES

We can also compare the étale homotopy type of the moduli stack \mathcal{A}_g with that of the moduli stack \mathcal{M}_g of algebraic curves of genus $g \geq 2$, which was determined by Oda [O]. The relation between the two algebraic stacks is given by the Torelli morphism (cf. [MFK, Sec. 7.4]). Given a family of algebraic curves $\pi : C \rightarrow U$ we can associate to it its Jacobian given as the relative Picard scheme $J(C/U) = \text{Pic}^0(C/U)$, which is an abelian scheme over U of relative dimension g . There exists a principal polarisation

$$\vartheta : J(C/U) \rightarrow \hat{J}(C/U) = \text{Pic}^0(J(C)/U)$$

induced by the Θ -divisor. This induces a functor

$$j(U) : \mathcal{M}_g(U) \rightarrow \mathcal{A}_g(U), \quad C/U \mapsto (J(C/U), \vartheta)$$

between the category of sections for every object U in the category of S -schemes (Sch/S). Therefore we get a morphism between algebraic stacks

$$j : \mathcal{M}_g \rightarrow \mathcal{A}_g.$$

Torelli's theorem (cf. [MFK]) basically states that for every algebraically closed field k the functor

$$j(\text{Spec}(k)) : \mathcal{M}_g(\text{Spec}(k)) \rightarrow \mathcal{A}_g(\text{Spec}(k))$$

is faithful. Oda [O] showed that the étale homotopy type of \mathcal{M}_g is given as the profinite completion of the classifying space of the mapping class group Map_g of compact Riemann surfaces of genus g , i.e. we have a weak homotopy equivalence of pro-simplicial sets

$$\{\mathcal{M}_g \otimes \bar{\mathbb{Q}}\}_{et}^\wedge \simeq K(\text{Map}_g, 1)^\wedge.$$

Therefore we get a morphism between pro-simplicial sets induced by the Torelli morphism j

$$\{\mathcal{M}_g \otimes \bar{\mathbb{Q}}\}_{et}^\wedge \rightarrow \{\mathcal{A}_g \otimes \bar{\mathbb{Q}}\}_{et}^\wedge$$

and for the étale fundamental groups a morphism between short exact sequences of profinite groups

$$\begin{array}{ccccccc}
 1 & \longrightarrow & (\mathrm{Map}_g)^\wedge & \longrightarrow & \pi_1^{et}(\mathcal{M}_g \otimes \mathbb{Q}, x) & \longrightarrow & \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}) \longrightarrow 1 \\
 & & \downarrow & & \downarrow & & \parallel \\
 1 & \longrightarrow & \mathrm{Sp}(2g, \mathbb{Z})^\wedge & \longrightarrow & \pi_1^{et}(\mathcal{A}_g \otimes \mathbb{Q}, j(x)) & \longrightarrow & \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}) \longrightarrow 1
 \end{array}$$

where the existence of the first short exact sequence is a direct consequence of the result of Oda.

We can also consider the moduli stack $\mathcal{M}_{g,[N]}$ of smooth algebraic curves of genus g endowed with a level N structure, where a level N structure on an algebraic curve C/S is a level N structure on the Jacobian variety $J(C/S)$ (cf. [MFK]).

We also have a Torelli morphism

$$j_N : \mathcal{M}_{g,[N]} \rightarrow \mathcal{A}_{g,[N]}$$

which is 2 : 1 on its image outside the hyperelliptic locus and it is ramified on the hyperelliptic locus (cf. [OS]).

For $N \geq 3$, the moduli stacks $\mathcal{M}_{g,[N]}$ are in fact a smooth quasi-projective schemes over $\mathrm{Spec}(\mathbb{Z}[N^{-1}])$ (cf. [S2]).

Its complex analytification $\mathcal{M}_{g,[N]}^{an}$ has the structure of a complex analytic orbifold and of a smooth complex manifold if $N \geq 3$. In fact, recall that the principal polarisation on the Jacobian of a compact Riemann surface C_g of genus g is given by the cup product on $H^1(C_g, \mathbb{Z})$. So if we denote as above by Map_g the mapping class group, we have a natural morphism $t : \mathrm{Map}_g \rightarrow \mathrm{Sp}(2g, \mathbb{Z})$, and also a morphism $t_N : \mathrm{Map}_g \rightarrow \mathrm{Sp}(2g, \mathbb{Z}/N\mathbb{Z})$. Denote by R_g and $R_g(N)$ the kernels of the morphisms t and t_N respectively. They both act properly discontinuously on the Teichmüller space \mathfrak{T}_g and the quotient $\mathfrak{Tor}_g = \mathfrak{T}_g/R_g$ is called the Torelli space, while the quotient $\mathfrak{T}_g/R_g(N)$ is the moduli space of compact Riemann surfaces of genus g with a level N structure. So we have a complex analytic orbifold structure on $\mathcal{M}_{g,[N]}^{an} = [\mathfrak{T}_g/R_g(N)]$. Moreover we have the following diagram for the Torelli morphisms:

$$\begin{array}{ccccc}
 \mathfrak{T}_g & & & & \\
 \downarrow & & & & \\
 \mathfrak{Tor}_g & \xrightarrow{j_{\mathfrak{Tor}}} & \mathfrak{H}_g & & \\
 \downarrow & & \downarrow & & \\
 \mathcal{M}_{g,[N]}^{an} & \xrightarrow{j_N} & \mathcal{A}_{g,[N]}^{an} & & \\
 \downarrow & & \downarrow & & \\
 \mathcal{M}_g^{an} & \xrightarrow{j} & \mathcal{A}_g^{an} & &
 \end{array}$$

where j is injective, while $j_{\mathfrak{Z}_{\text{or}}}$ and j_N are $2 : 1$ on their respective images and ramified on the hyperelliptic locus (cf. [OS]).

It follows in the same way as Oda's result for \mathcal{M}_g that the étale homotopy type of the moduli stack $\mathcal{M}_{g,[N]}$ is given as the profinite completion of the classifying space of the discrete group $R_g(N)$, i.e. we have a weak homotopy equivalence of pro-simplicial sets

$$\{\mathcal{M}_{g,[N]} \otimes \bar{\mathbb{Q}}\}_{et}^\wedge \simeq K(R_g(N), 1)^\wedge.$$

This follows readily from the fact that $\mathcal{M}_{g,[N]}^{an} = [\mathfrak{T}_g/R_g(N)]$ and that the Teichmüller space \mathfrak{T}_g is again contractible by using the comparison theorem for homotopy types (cf. Theorem 2.6).

And so we get again a morphism between pro-simplicial sets induced by the Torelli morphism j_N , which preserves the level N structures

$$\{\mathcal{M}_{g,[N]} \otimes \bar{\mathbb{Q}}\}_{et}^\wedge \rightarrow \{\mathcal{A}_{g,[N]} \otimes \bar{\mathbb{Q}}\}_{et}^\wedge$$

and for the étale fundamental groups a morphism between short exact sequences of profinite groups

$$\begin{array}{ccccccc} 1 & \longrightarrow & R_g(N)^\wedge & \longrightarrow & \pi_1^{et}(\mathcal{M}_{g,[N]} \otimes \mathbb{Q}, x) & \longrightarrow & \text{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}) \longrightarrow 1 \\ & & \downarrow & & \downarrow & & \parallel \\ 1 & \longrightarrow & \Gamma_{2g}(N)^\wedge & \longrightarrow & \pi_1^{et}(\mathcal{A}_{g,[N]} \otimes \mathbb{Q}, j_N(x)) & \longrightarrow & \text{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}) \longrightarrow 1 \end{array}$$

using the calculations of étale homotopy types of the moduli stacks and the short exact sequences for the respective étale fundamental groups.

These observations now allow to study algebro-geometric properties of the Torelli morphisms and questions related to the Schottky problem of characterising the locus of Jacobians among principally polarised abelian varieties in terms of étale homotopy types of the associated morphism between algebraic stacks. This will be a theme in a follow-up article.

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